

MATHEMATICAL KNOWLEDGE FOR TEACHING PROOF: EVIDENCE FROM AND IMPLICATIONS FOR PROFESSIONAL DEVELOPMENT

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Research suggests that the enhanced role of proof in mathematics classrooms presented in current standards and reform policy poses great challenges for teachers and will require substantial teacher learning. However, to date there is little research detailing what mathematical knowledge might be useful for teaching proof or how professional development might afford such learning. This paper presents a framework for Mathematical Knowledge for Teaching Proof that couples research on justification and proof in mathematics and mathematics education with Ball and colleagues' (2008) conceptualization of Mathematical Knowledge for Teaching (MKT). An empirical study of teachers' proof-related activity in professional development demonstrates the utility of this framework and provides further insights into mathematical knowledge for teaching proof.

Few can dispute the increased attention that reasoning and proof have been afforded in mathematics education research and practice (Hanna & deVilliers, 2008; National Council of Teachers of Mathematics, 2009). Both NCTM (2009) and the Common Core State Standards (2010) recommend that all students learn to construct and evaluate mathematical arguments and demonstrate a variety of reasoning and proof techniques. More notably, these documents call for an enhanced notion of proof that elevates proof beyond a topic of study in advanced mathematics courses, to a tool for learning mathematics at all levels.

However, the research summarizing students' difficulties constructing proofs and their reliance on authority or empirical arguments to justify a general claim suggests that these recommendations are not easily accomplished (Harel & Sowder 2007; Healy & Hoyles, 2000). Similar research has shown that teachers often rely on ritualistic aspects of proof and fail to distinguish between non-proof arguments and valid deductive reasoning when both constructing and evaluating proofs (Martin & Harel, 1989; Knuth, 2002; Simon & Blume, 1996). Moreover, recent classroom studies show that teachers struggle to support students even when using a curriculum designed to promote reasoning and proof (Bieda, 2010). For example, teachers' methods of grouping and sequencing proof strategies or inability to follow up on students' ideas often inhibit students' ability to construct a valid proof (Martin & McCrone, 2003; Bieda, 2010). Other studies have linked teachers' methods of questioning and providing examples to students' development of authoritarian or empirical proof schemes (Harel & Rabin, 2010).

Clearly, supporting all students to learn about and engage in mathematical proof poses great challenges for teachers and will require substantial teacher learning. However, to date there is little research detailing what mathematical knowledge might be useful for teaching proof or how professional development might afford such learning.

Purpose

The purpose of the larger study was to detail Mathematical Knowledge for Teaching Proof (MKT for proof) and to investigate MKT for proof in professional development (PD). To advance the construct of mathematical knowledge for teaching proof, research on proof and mathematical knowledge for teaching was coordinated with an empirical study of teachers'

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engagement with proof in professional development. Specifically, the study aimed to address the following questions:

1. What does research suggest teachers need to know about proof and proving that would be useful for their work with students?
2. What mathematical knowledge for teaching proof is evidenced in professional development focused on justification and proof?

This paper describes how an initial framework for MKT for proof was developed based on an extensive literature review. A sample of findings from one segment of the larger investigation of teachers' proof activity in PD is then provided to further highlight elements within this knowledge framework.

Theoretical Considerations

Two ideas underlie the conceptualization of mathematical knowledge for teaching proof developed through this study: (i) an enhanced notion of proof as a means to support mathematical understanding at all levels and (ii) the important role of teachers' mathematical knowledge-in-use (knowledge accessed when engaging in acts of teaching) for considering a knowledge base for teaching.

Recognizing the critical importance of proof in learning mathematics, recent reform efforts call for proof to be a regular and ongoing part of students' K-12 mathematics experiences (NCTM, 2000). To support proof as a sense-making activity, there is a need for a definition that promotes a consistent meaning of proof throughout the grades. For this reason, Stylianides' (2007) definition of proof was adopted for this study. Put simply, proof refers to a mathematical argument that is based on accepted statements, valid modes of argumentation, and representations that are known by or are within the conceptual reach of the classroom community. Further, to identify and describe the mathematical knowledge teachers need to engage students in practices consistent with this reform-oriented view of proof, the entire range of activity associated with proving must be considered. These proving activities include identifying patterns, making conjectures, testing examples, and providing non-proof arguments, as well as constructing proofs.

Secondly, a situative perspective on learning informs both the conceptualization of mathematical knowledge for teaching proof and the ways in which it might be investigated in PD settings. This perspective takes seriously the ways in which mathematical knowledge is attuned to the specific demands of teaching (Adler & Davis, 2006). Defining mathematical knowledge for teaching proof thus involves consideration of what, when, and how knowledge is required and used to meet the demands of the classroom. Central to a situative perspective is the recognition that cognition is situated in particular contexts, social in nature, and distributed across the individual, other persons, and tools (Putnam & Borko, 2000). Accordingly, mathematical knowledge evidenced in interaction as teachers participated in proof-related activities was the focus of the qualitative analysis.

Methods

Developing MKT for Proof Framework

To develop a framework for MKT for proof, research on mathematical proof, students' difficulties with proof, and classroom studies of teaching practices related to proof were reviewed, coordinated and synthesized. Proof ideas drawn from the literature were categorized using four domains of MKT introduced by Ball and colleagues (2008). Tables 1 and 2 below provide a snapshot of elements included in this MKT for proof framework. Common Content

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Knowledge (CCK), the subject matter knowledge held in common with others who use mathematics, includes the ability to construct a valid proof as well as common understandings of the nature of proof. Elements within CCK reflect essential proof knowledge and skills desired of students but also directly address findings that teachers often attend to the form of proof rather than substance and fail to see the generality in an argument (Dreyfus & Hadas, 1987; Martin & Harel, 1989).

For their work with students, teachers need more than CCK. Fundamental understandings of proof must be developed, “unpacked,” and explicitly connected to the work of teaching. This Specialized Content Knowledge (SCK) is pure mathematical knowledge uniquely needed for teaching proof. While mathematicians may know how proofs depend on previously known definitions, to deal with issues that arise in the classroom, teachers also need to know a range of possible definitions and understand how an argument might look different depending on the definitions that are accepted. To guide their instruction, teachers also need to understand the relationship between the proving task and valid or efficient methods of proof such tasks evoke (Stylianides & Ball, 2008). Thus, knowing that a claim about a finite number of cases can be verified through a systematic list, but that a generic example or deductive argument would be needed to prove a more general claim would be considered SCK.

Table 1. Subject Matter Knowledge for Teaching Proof

Common Content Knowledge	Specialized Content Knowledge
<p>Ability to construct valid proof</p> <p>Use definitions, theorems to build a logical progression of statements</p> <p>Analyze situations by cases & use counterexamples</p> <p>Nature of Proof</p> <p><i>A theorem has no exceptions</i></p> <p><i>A proof must be general</i></p> <p>The validity of a proof depends on its logic</p>	<p>Explicit Understanding of Proof Components</p> <ul style="list-style-type: none"> •Accepted Statements: range of definitions or key theorems, role of language & defined terms •<i>Modes of Argumentation: recognize range of valid methods, distinguish between empirical and deductive arguments</i> •<i>Modes of Representation: variety of visual, symbolic, verbal methods to express general argument</i> <p>Relationship between proving tasks & activity</p>

Elements within pedagogical content knowledge arose from studies documenting students’ typical responses when asked to produce or evaluate mathematical proofs as well as classroom studies that illustrated where proof instruction fell short. Although these studies did not specifically explore teachers’ knowledge, inferences can be made regarding knowledge or resources teachers might have drawn upon to support students’ understanding of proof. As indicated in table 2, Knowledge of Content and Students (KCS) includes detailed knowledge of students’ thinking as well as attention to the proof-related resources available to students. Knowledge of Content and Teaching (KCT) intertwines knowledge of proof from the other domains with methods of representing or drawing out key proof ideas in classroom instruction. For example, knowing that students typically rely on authority or empirical justification (Harel & Sowder, 2007) guides instructional decisions and questions or examples teachers may use to encourage students’ progression toward deductive arguments.

Clearly, no one idea within MKT for proof is unique, nor is this meant to be a comprehensive

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list of everything a teacher needs to know. However, this framework provides a rare synthesis of research and begins to detail the mathematical knowledge of proof that would be useful for teachers' work with students. As described next, delineating knowledge of proof in this way provided a common analytic tool to make sense of teachers' proof work across different professional development activities.

Table 2. Pedagogical Content Knowledge for Teaching Proof

Knowledge of Content and Students	Knowledge of Content and Teaching
<p>Detailed knowledge of student thinking</p> <ul style="list-style-type: none"> • Explicit knowledge of proof schemes • Recognizing characteristics of external, empirical, and deductive proof schemes • Students' tendency to rely on authority or empirical examples • <i>Progression from inductive to deductive proof</i> <p>Developmental aspects of proof</p> <p>Ability to size up arguments in terms of:</p> <ul style="list-style-type: none"> • Definitions & statements available to students • <i>Representations within students conceptual reach</i> • <i>Forms of argumentation appropriate for students' level</i> 	<p>Relationship between instruction and proof schemes</p> <p>Questioning strategies</p> <ul style="list-style-type: none"> • To elicit justification beyond procedures • <i>To encourage thinking about general case</i> <p>Pivotal examples or counterexamples</p> <ul style="list-style-type: none"> • To extend, bridge or scaffold thinking • To focus on key proof ideas <p>Explicit knowledge of proof connections</p> <ul style="list-style-type: none"> • <i>Linking visual, symbolic or verbal proofs of same concept or theorem</i> • Comparing proofs in terms of accepted definitions and argument structure • <i>Lifting general argument from numerical example or specific diagram</i>

Investigating MKT for Proof in Professional Development

The empirical study used to test and refine this framework, drew upon an existing research project designed to support teacher leaders' understanding and facilitation of mathematically rich PD. Although teachers' knowledge of proof was not the central project purpose, the focus on generalizing patterns and number concepts and explicit attention to norms for justification and explanation made the project an ideal site to investigate MKT for proof. Within this project, K-12 teacher-leaders participated in either three or four two-day seminars during the school year. During these seminars, teachers worked on a number of mathematics tasks, analyzed videocases of PD sessions centered on the same mathematical tasks, and engaged in additional activities designed to connect these experiences to their own practice.

Given the goal of detailing MKT for proof, all math tasks, discussion prompts, video data logs, and seminar agendas were reviewed to select tasks in which teachers were asked to make conjectures, prove claims, and engage in explicit conversations about proof in the context of teaching. Two mathematical tasks, Consecutive Sums and Halving and Doubling, were selected. This paper reports on teachers' seminar work associated with the Halving and Doubling task. In particular, video data of two small groups and one whole group discussion for each of eight activities directly related to the Halving and Doubling task were analyzed. These activities included teachers 1) doing the math task; 2) viewing PD video to compare two teacher justifications; 3) viewing video to consider an additional conjecture 4) discussing MKT in relation to video; 5) revisiting video to consider facilitation of the task; 6) planning PD session using task; 7) discussing justification article (Lannin, 2005); 8) reflecting on their own PD enactment of task with colleagues.

Analysis

To address the second research question, what MKT for proof is evidenced in PD, all

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Halving and Doubling video was reviewed to develop a coding scheme that would capture the nature of teachers' conversations in relation to proof. Initial coding indicated when teachers referred to visual or algebraic models, tested specific examples, explicitly used the terms "conjecture," "generalization," or "proof," as well as when discussions moved to student or teaching concerns. Studiocode (2010), a video analysis software, was used to facilitate the coding and analysis of each of the small group and whole group discussions. The data matrix features of Studiocode then allowed for both in-depth analysis of a proof topic as well as an indication of how and if this proof idea was evidenced across the range of seminar activities. For example, all instances of teachers using the word "proof" or attempting to understand a generic example could be gathered and viewed in succession to facilitate more focused coding. As patterns emerged from this analysis, integrative memos (Emerson, Fretz & Shaw, 1995) were written to summarize elements of proof consistent across teachers' discussion. It was only at this stage that proof ideas emerging from seminar activity were mapped back to the MKT framework.

Results

To illustrate this mapping, findings from analysis of teachers' seminar activity provided here are organized around the four domains of teacher knowledge: CCK, SCK, KCS, and KCT, within the MKT for proof framework. Elements in tables 1 and 2 corresponding to these reported findings have been italicized to further highlight the relationship between the teachers' proof activity within Halving and Doubling and categories of MKT for proof.

CCK: Nature of Proof

Teachers Recognized the Need for Proof to be General

Across all small and whole group conversations teachers used phrases that clearly related to the general nature of proof. Teachers recognized flaws in arguments that were based on a few specific cases or did not explicitly demonstrate that the conjecture was valid for *any* numbers. When evaluating ways of verifying the conjecture, comments such as the following were typical:

"I like the algebraic model that you came up with and then also the area model because you can show that for *any* string of numbers if you half it and then add to the end - that is essentially what you are doing" (sg1 – doing math)

"In the rectangle the use of a and b *is an attempt to generalize* and regardless of what it looks like visually, the a and the b represent variables and therefore *any case*." (wg - comparing videocase teacher work)

Teachers both pressed for generalization in their own mathematical work and recognized attempts the teachers in the PD videocase made toward presenting general arguments.

SCK: Explicit Understanding of Proof Components

Teachers Connected a Variety of Visual, Symbolic, and Verbal Methods Used to Express Argument

Teachers went beyond noting that a proof must be general to explore a variety of visual, symbolic, and verbal methods. In their own mathematical work small groups explored the problem through the use of numerical tables, array models, and algebraic expressions. More importantly, as illustrated in the two previous quotes, teachers considered how those representations could be used to express a general argument. As teachers continued to evaluate and connect various methods of verifying the conjecture, they grappled with ways in which a visual diagram might move beyond a specific case to provide a generic example or connect to an

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algebraic proof. Discussions such as the ones prompted by the teacher comments below firmly situated teachers in the realm of SCK for proof.

“I don't know that your method lends itself to an algebraic proof or justification because it is only showing concrete, discrete numbers its not showing me how I would connect a variable standing for any number...”

So what would we need to add to my model, my proof to meet that criteria?” (sg2 – role playing after video viewing)

KCS: Developmental Aspects of Proof

Acceptable Proof Depends on Forms of Argumentation and Representations Appropriate to Grade Level

In both teachers' discussions and actions it was evident that when considering what might be an acceptable proof of the Halving and Doubling conjecture, teachers considered what knowledge and skills students at a given grade level typically possess. Teachers made explicit statements that what constitutes a proof might depend on the grade level you were teaching and “where your students are at.” In terms of representations, teachers asked questions about whether students had a grasp of variable, or were comfortable with area or array models to represent multiplication. And in considering forms of argumentation that might be accessible to students, teachers acknowledged students' ability to see generality in the array diagrams but wondered if elementary students would be capable of constructing a deductive argument.

KCT: Explicit Knowledge of Proof Connections

Teachers Discuss Scaffolding from Examples or Visual Proof to Symbolic Representations by Capitalizing on a Key Idea or Generality

When discussing representations and forms of argument students may or may not understand, teachers often described specific teaching strategies they may employ. For example, in this statement below, a teacher is imagining how he might lead a class discussion of the problem by exploring patterns across numerical examples and then moving students toward the use of variables to make a more general argument:

“I might bring up multiple examples, then say ‘what I'm hearing you say is no matter what numbers we use, the operation stays the same so let's put a letter in’” (sg1 - doing the math)

Teachers further explored ways in which visual array or area models might be introduced to make connections between a specific example and a more general argument using variables. Importantly, teachers were able to bring out key mathematical ideas underlying the generalization (reciprocal properties, conservation of area, commutativity, etc.) when suggesting strategies to move students from example-based proofs toward more general arguments.

Discussion

Two findings from the Halving and Doubling teacher discussions in particular shed light on previous research regarding teachers' understanding of proof and what MKT for proof might entail. First, it is worth repeating that although teachers recognized essential features of a valid proof, this was not always evidenced in their acceptance of a proof. For example, as highlighted earlier, teachers made clear statements that proof must be general. It must work for *any* number and testing only specific numbers was not enough to prove the general claim. And yet teachers also discussed how in 3rd grade, it might be okay to use specific numbers in an array diagram, or how they might begin instruction by having students look for patterns in a table. Or, teachers might follow up on a valid general argument by asking how it would work for fractions, odd numbers, or negative numbers. In other words, while empirical justification clearly was not

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enough proof, teachers saw pedagogical value in posing questions about different classes of numbers or in exploring visual representations. As teachers, they saw the value of testing examples or using specific visual representations for helping students make sense of the mathematics involved and providing a foundation for algebraic proof. This finding makes us step back and reconsider the research on what teachers “understand” about proof. While previous research has suggested that teachers fail to recognize the generality of proof, here we see a sophisticated understanding of generality that is clearly tied to the context of teachers’ work.

A second, and closely related finding, is that teachers moved fluently across the four domains of knowledge. Teacher talk within all eight seminar activities took up multiple dimensions of the MKT for proof framework. Interjections about what representations students have access to or what numbers to use in examples were intermingled with explanations of teachers’ own proof constructions. Teachers’ genuine questions about generic examples or the difference between inductive and deductive reasoning led directly to issues of students and of teaching. This finding further highlights the situated nature of teachers’ knowledge and calls to question PD models focusing on either mathematics or pedagogy. As discussed next, this has implications for how the mathematics education field might begin to detail, research, and develop MKT.

Implications

First and foremost, this study contributes to the growing research on teacher knowledge by detailing critical aspects of mathematical knowledge of proof that support teachers’ work with students. Recent research addressing teacher learning has identified *mathematical knowledge for teaching* as a key construct in teacher education (Ball, Lubienski & Mewborn, 2001). However, most attempts to specify mathematical knowledge for teaching have focused on elementary content. This investigation extends the work to mathematical knowledge useful for promoting proof, a key mathematical practice, K-12.

Further, this research provides a common framework for analyzing MKT for proof in both classroom and PD settings. The framework supports research on the teaching of proof by serving as an analytic tool to identify mathematical resources teachers access as they engage students in proof activity. The framework can also be used to make sense of teachers’ proof-related activity in PD by connecting specific teacher activity such as constructing proofs versus comparing proofs presented in a videocase to the domains of teacher knowledge evidenced in discussions. This work supports further research on what “effective” PD to develop teachers MKT for proof might entail.

Finally, the findings related to teachers’ fluent movement across knowledge domains help advance theory about the complex relationship between subject matter and pedagogical content knowledge. Investigation of conversations in which teachers move fluidly from talking about the mathematical ideas embedded in a proof to thinking about students and teaching can help the field better understand those connections between the four domains of knowledge - CCK, SCK, KCS, and KCT. This close relationship has implications for both researching and designing activities that might support teachers’ development of MKT.

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